AN EXTENDED KINEMATIC EQUATION

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Abstract
This brief technical review article covers the original 1960 kinematic equation universally adopted 1965 to regulate yellow change intervals and how an extended version was developed 2015 to include vehicle deceleration required for turning maneuvers using basic science, illustrations of the kinematic equations, and vehicle dynamics test data.

Introduction
The yellow traffic signal was first added between the green and red intervals in 1920. Its primary function is to warn a driver in a vehicle approaching an intersection that the red signal and change in the right-of-way are imminent. Seemingly, the timing of the yellow indication appears straightforward. However, determining the illumination interval is quite intricate since it is part of a complex system, including physical and human-made laws, technology, and human behaviors that all must be compatible.

In 1960, Denos Gazis, Robert Herman, and Alexei A. Maradudin (GHM) provided a scientific solution for the yellow change interval in their paper, "The Problem of the Amber Signal Light in Traffic Flow" [1]. Using the science developed by Sir Isaac Newton called calculus-based physics or kinematics, GHM analyzed the optional STOP or GO vehicle motions approaching an intersection. The problem GHM solved and eliminated was an area in the roadway known as the "dilemma zone," where a driver-vehicle complex faced with a yellow indication could neither STOP before an intersection safely and comfortably nor GO safely and legally through.

GHM's solution to regulate a yellow change interval first appeared in 1965 [2], and it has become known as the kinematic equation. However, GHM's yellow interval is limited to vehicle motion in one spatial dimension at a constant velocity or in other words, straight-line travel through level intersections, not including deceleration required before turning maneuvers. This article covers GHM's solution and how an extended version was developed to include vehicle deceleration. The presented variables are defined in the appendix.

GHM's STOP or GO Solutions
The foundation of GHM's solutions is a minimum DISTANCE for a driver-vehicle complex to STOP termed the "critical distance" (x_C). This distance is composed of two parts, a distance to perceive and react to the yellow onset and a minimum braking distance. These distances are defined by the maximum approach velocity (v_0), a maximum allocated perception-reaction time (t_{PR}) and a maximum safe and comfortable uniform deceleration (a_{max}) to STOP, as shown in Equation (1):

$$x_C = v_0 t_{PR} + \frac{v_0^2}{2a_{max}} \tag{1}$$

GHM's solution to eliminate the dilemma zone is a minimum TIME for a driver-vehicle complex to GO the optional minimum STOP distance and through an intersection without the need to accelerate. Hence, the solution describes a vehicle with length (L) moving at the maximum constant approach velocity (v_0) and traversing the critical distance (x_C), through and exit a level intersection with an effective width (w). Mathematically, GHM converted these distances to a minimum restrictive [3] yellow change interval (Y_R) for a vehicle moving at the maximum constant velocity (v_0), as follows:

$$Y_R \geq \frac{x_C + w + L}{v_0} \tag{2}$$

Solving Equation (2) by inserting Equation (1) yields GHM's original minimum restrictive yellow change interval (Y_R) as presented in Equation (3) and illustrated in figure 1:

$$Y_R \geq t_{PR} + \frac{v_0}{2a_{max}} + \frac{w + L}{v_0} \tag{3}$$

GHM's minimum permissive [3] yellow change interval (Y_P) is the well-known kinematic equation. It originates from Equation (3) and its first two terms as presented in Equation (4) and illustrated in Figure 1:

$$Y_P \geq t_{PR} + \frac{v_0}{2a_{max}} \tag{4}$$

Figure 1 illustrates GHM's minimum STOP and GO Equations (1), (3), and (4). They are plotted together in a velocity vs. distance graph showing the optional STOP or GO uniform (constant or average) vehicle motions for the restrictive (Y_R) (warning and clearing) and the permissive (Y_P) (only warning) yellow timing policies.

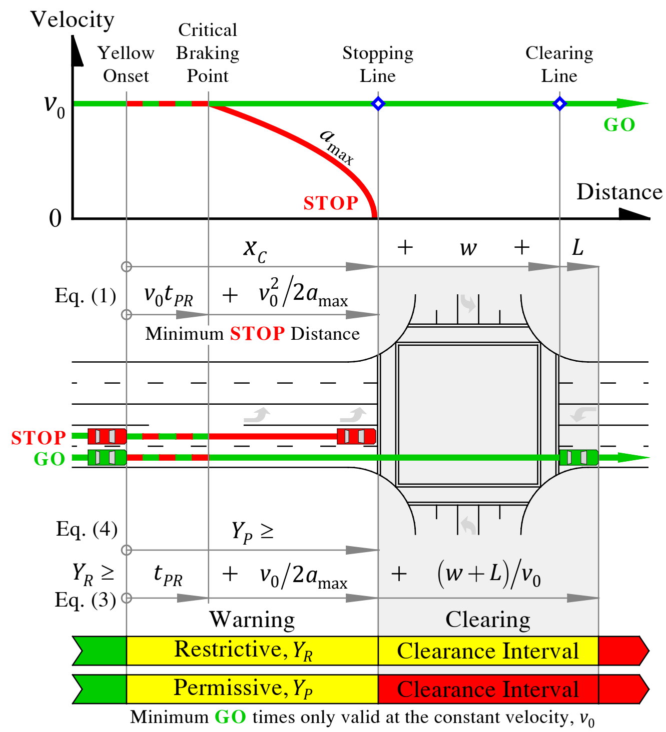

Figure 1 - GHM's minimum STOP and GO equations plotted and referenced a typical intersection

The model in this article uses the most common permissive timing policy, which function is to warn a driver that the red signal and change in the right-of-way are imminent, thus endorsing a uniform international timing standard.

The Kinematic Equation

GHM's solution first appeared in the 1965 ITE Traffic Engineering Handbook by Baerwald [2]. However, the adopted equations and variables lost GHM's specified limits and never found in subsequently published revisions. The LIMITS are critical in understanding GHM's timing model and how to implement the input variables and tolerances correctly. Besides, it also incorrectly claimed that GHM's time to GO, Equation (4), calculates the time to STOP, which is still a common misconception. Even Gazis and Herman, together with Chiu Liu, noted this error in their 1996 review [4] where they explained the time to STOP:

> *"The factor 2 in the denominator should be eliminated since it corresponds to covering the stopping distance at constant speed."*

Hence, the minimum time required for a vehicle to come to a safe and comfortable STOP and that also corrects Baerwald's definition of an "excessive yellow" or a maximum yellow interval (Y_{max}) regardless of signal timing policy, is Equation (4) without the "2" in the denominator and assuming a minimum perception-reaction time (t_{PR}) as follows:

$$Y_{max} = t_{PR} + \frac{v_0}{a_{max}} \tag{5}$$

Equation (5), the maximum yellow interval (Y_{max}) is a critical limit to avoid having drivers stopped at the stopping line still faced with the yellow indication.

Handbooks [2] [3] [5] commonly neglect to show the kinematic equation's limits, such as the maximum deceleration (a_{max}) or the minimum yellow duration, shown with a (Y_{min}) or a "greater than sign." For example, applying the equal sign of Equation (4) calculates a minimum permissive yellow interval (Y_P) and it results in a single-point STOP or GO solution at the maximum constant approach velocity (v_0) with zero tolerance for human errors.

To illustrate the limited function of the kinematic equation as a minimum, Figure 2 takes help from GHM's original paper [1], where they applied a standard kinematic distance (x) equation of accelerated (a) uniform motion in this general form:

$$x = vt + \frac{at^2}{2} \tag{6}$$

The second equation of GHM's paper uses Equation (6) to describe accelerated vehicle motion across the minimum braking distance. However, GHM's minimum GO time solution traverses the braking distance at the maximum constant velocity (v_0), which means zero acceleration ($a = 0$). This eliminates the last term of Equation (6) thus defining ($v = v_0$) and ($t = (Y_P - t_{PR})$) from Equation (4). Yet, to include accelerated vehicle motion, GHM's variable ($a = a_1$) is used along with their perception-reaction distance ($v_0 t_{PR}$) to reference the full critical distance (x_C) producing the following:

$$x = v_0 t_{PR} + v_0(Y_P - t_{PR}) + \frac{a_1(Y_P - t_{PR})^2}{2} \tag{7}$$

Substituting ($Y_P - t_{PR}$) with the last term from Equation (4) and further simplification of Equation (7) yields Equation (8) where the first two terms are the critical distance (x_C):

$$x = v_0 t_{PR} + \frac{v_0^2}{2a_{max}} + \frac{a_1}{2}\left(\frac{v_0}{2a_{max}}\right)^2 \tag{8}$$

Equation (8) calculates the traveled distance (x) to either STOP or GO at changing vehicle accelerations ($\pm a_1$) across the minimum braking distance during a minimum permissive yellow interval (Y_P) from Equation (4):

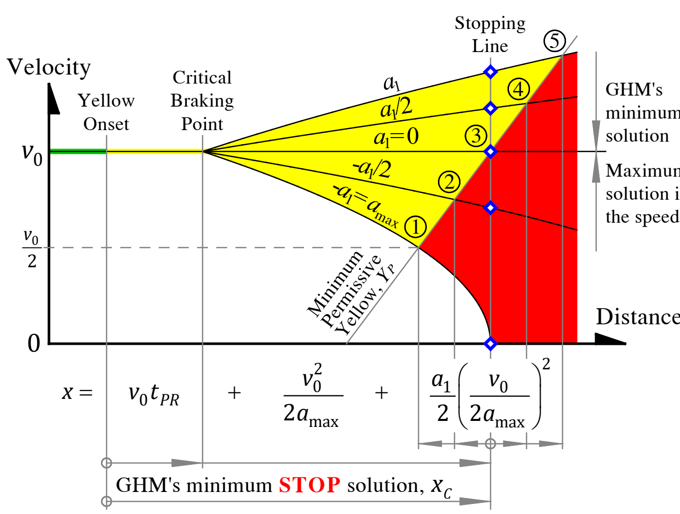

Figure 2 - The kinematic equation as a minimum plotted with accelerated uniform vehicle motions

Figure 2 shows a driver in a vehicle approaching at the maximum speed limit (v_0), faced with the yellow onset at the critical distance (x_C) and responding at the critical braking point with optional accelerated vehicle motions:

① Maximum deceleration ($-a_1 = a_{max}$) to STOP, defining GHM's critical distance (x_C).
② Deceleration ($a_{max} < -a_1 < 0$) before entering the intersection, the vehicle runs a red light.

③ Zero acceleration ($a_1 = 0$), constant velocity. The vehicle travels the exact GHM's critical distance (x_C).
④ Acceleration ($a_1 > 0$) towards the intersection. The vehicle enters during a yellow indication, which is typical driver behavior. However, any acceleration exceeds the speed limit defined by the maximum approach velocity (v_0).
⑤ Same as number 4.

The optional accelerated vehicle motions in Figure 2 show that the kinematic equation as a minimum is limited to STOP at maximum deceleration or GO at constant velocity, a binary solution with zero tolerance for human errors.

Vehicle Motion Illustrated in Time

In a velocity vs. time graph, uniform (constant) vehicle motions such as velocity or deceleration are linearly plotted, and the traveled distances are as per Newton's calculus-based physics the areas between a motion plot and the time-axis. Hence, velocity vs. time graphs allow a combined representation of both time and distance, such as GHM's permissive minimum GO time (Y_P) and STOP distance, the critical distance (x_C), as presented in Figure 3.

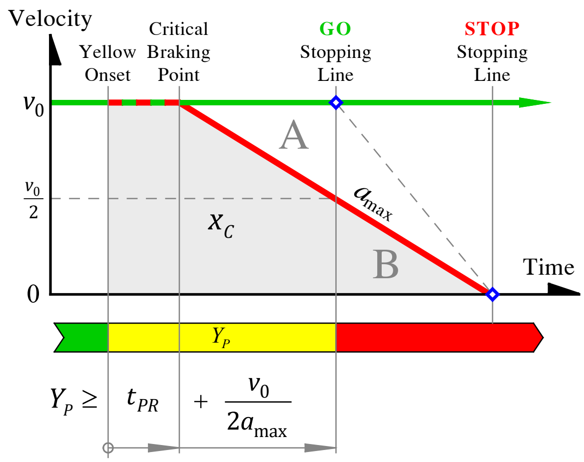

Figure 3 - GHM's permissive minimum STOP and GO solutions combined in a velocity vs. time graph

In Figure 3, GHM's STOP solution with its maximum safe and comfortable constant deceleration (a_{max}) is linearly represented in time and its STOP distance, the critical distance (x_C) from the yellow onset to the stopping line, is the shaded grey area between its STOP motion plot and the time-axis. GHM's minimum permissive GO solution to eliminate the dilemma zone is describing a vehicle traversing the same critical distance at constant velocity since the areas marked A and B are equal, a single point solution.

The Extended Kinematic Equation

The extended kinematic equation is an adjustable linear timing model adaptable to any approach lane, including turning lanes where vehicle deceleration is required before making safe and comfortable turning maneuvers. The extended equation is derived from GHM's original solutions, and it was discovered studying motion plots of GHM's optional STOP or GO solutions, as in Figure 3, while investigating turning maneuvers.

The key to the discovery in 2015 was the identification of the source of GHM's solution, which is their minimum braking distance defined in their paper's very first equation. In turn, the minimum braking distance is defined by the maximum uniform safe and comfortable deceleration (a_{max}). Hence, the maximum deceleration trajectory ending before the stopping line is the actual STOP or GO boundary.

The extended kinematic model utilizes GHM's critical distance (x_C) and its maximum motion limits and STOP trajectory, but the vehicle does not STOP. Instead, the vehicle departs the STOP trajectory to enter the intersection at the new maximum uniform (constant) intermediate/entry velocity (v_1) that is equal to or less than the maximum approach velocity (v_0). The new velocity variable (v_1) adds a kinematic section within GHM's critical distance which produces an extended model with three successive sections describing the following driver-vehicle activities:

① Perception-Reaction - A time (t_{PR}) to perceived and react to the yellow signal onset at constant velocity (v_0).

② Deceleration - The maximum uniform (constant) safe and comfortable deceleration trajectory (a_{max}).

③ Entering - The new maximum uniform (constant) intermediate/entry velocity variable (v_1).

Figure 4 illustrates the extended model's three sections which yield the minimum extended permissive yellow change interval (Y_{EP}) Equation (9):

$$Y_{EP} \geq \overset{①}{t_{PR}} + \overset{②}{\frac{(v_0 - v_1)}{a_{max}}} + \overset{③}{\frac{½v_1}{a_{max}}} \tag{9}$$

Further simplification of Equation (9) produces the final extended kinematic equation shown in Equation (10):

$$Y_{EP} \geq t_{PR} + \frac{v_0 - ½v_1}{a_{max}} \tag{10}$$

Equation (10) is valid for ($v_0 \geq v_1 > 0$) and if (v_0) is the speed limit it provides an extended solution including deceleration with an entry velocity range from (v_0) to (v_1). The following are entry velocity (v_1) examples:

$v_1 = v_0$ Constant velocity yields GHM's minimum permissive yellow interval, Equation (4).

$v_1 = 0$ Zero end velocity yields the minimum time it takes to STOP defined by Equation (5), the maximum yellow interval (Y_{max}) which assumes a minimum perception-reaction time (t_{PR}).

$v_1 > 0$ Entry velocities are always greater than zero yielding yellow intervals less than (Y_{max}).

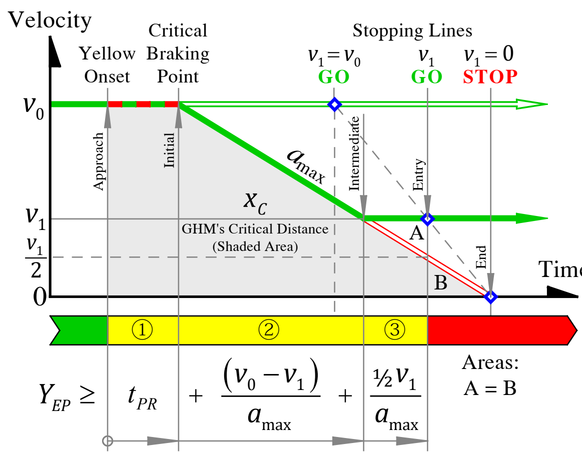

Figure 4 - Kinematic model adding vehicle deceleration within GHM's critical distance (x_C)

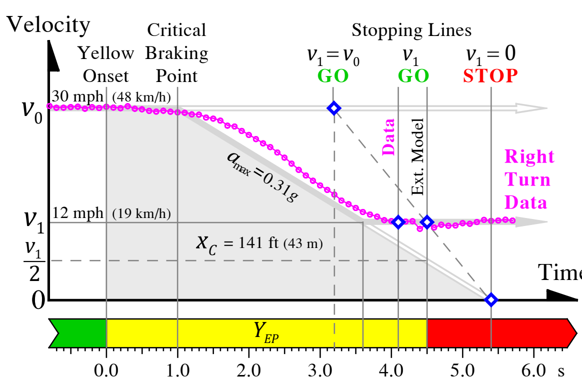

Figure 5 - The extended kinematic model with instantaneous vehicle dynamics right turn data

Figure 5 shows recorded vehicle motion data traversing GHM's critical distance (x_C) (shaded grey area) before a right turn, sampled at 10 Hz using a Racelogic Video VBOX Lite GPS data logger. The data verifies that a vehicle decelerating within the critical distance before making a turning maneuver follows the extended kinematic model for Equation (10) closely, and it allows the driver-vehicle complex to enter legally on a yellow signal indication.

Summary
This article has shown the functions and limits of GHM's original solution and presented an extended kinematic version expanding upon GHM's logic. The result is a universal solution applicable to any intersection's approach, including turning lanes. It is a vehicle's motion and its path through an intersection that ultimately determines the time necessary to traverse the intersection, which becomes the principal source of a yellow change interval. The extended kinematic model adds vehicle deceleration within the critical distance through the introduction of a new entry velocity variable. Thus, providing an adjustable minimum timing solution applicable for any approach lane, maneuver, or type of vehicle, including autonomous vehicles, traversing through level intersections.

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APPENDIX

Definition of Variables

x_C = GHM's critical distance - the minimum safe and comfortable stopping distance, (ft or m)
x = Traveled distance in one spatial dimension, (ft or m)
w = The effective width of the intersection, (ft or m)
L = Length of the vehicle, (ft or m)
Y_R = Minimum duration of the restrictive yellow signal indication, (s)
Y_P = Minimum duration of the permissive yellow signal indication, (s)
Y_{EP} = Minimum duration of the extended permissive yellow signal indication, (s)
Y_{max} = Maximum duration of the yellow signal indication, (s)
t_{PR} = Maximum allocated driver-vehicle perception-reaction time, (s)
t = Time, (s)
v_0 = Maximum uniform initial/approach velocity, (ft/s or m/s)
v_1 = Maximum uniform intermediate/entry velocity, (ft/s or m/s)
v = Uniform velocity, (ft/s or m/s)
a_{max} = Maximum uniform driver-vehicle safe and comfortable deceleration, (ft/s^2 or m/s^2)
a_1 = GHM's uniform acceleration variable, (ft/s^2 or m/s^2)
a = Uniform acceleration, (ft/s^2 or m/s^2)

References

[1] D. Gazis, R. Herman and A. A. Maradudin, "The Problem of the Amber Signal Light in Traffic Flow," *Operations Research,* vol. 8, no. 1, pp. 112-132, 1960.

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[4] C. Liu, R. Herman and D. C. Gazis, "A Review of the Yellow Interval Dilemma," *Transportation Research Part A: Policy and Practice,* vol. 30, no. 5, pp. 333-348, 1996.

[5] J. H. Kell and I. J. Fullerton, "Phase-Change Interval," in *Manual of Traffic Signal Design*, Englewood Cliffs, NJ 07632, Prentice-Hall, 1982, pp. 141-144.

In memory of
Marianne & Roland Järlström, Sigge Wennlöf, Lois & David Hodge,
Don Peyton, Gordon Long, and Chris Strahm.