Derivation of the Yellow Change Interval Formula

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The yellow change interval formula traffic engineers use to set yellow light durations originated from a paper written by Denos Gazis, Robert Herman and Alexei Maradudin[1]. The paper is called *The Problem with the Amber Signal in Traffic Flow.* They wrote the paper in 1959 as part of their work with General Motors.

The Institute of Transportation Engineers (ITE) adopted this formula in 1965[2]. ITE modified this formula in 1982 to include the effect of the earth's gravity upon a vehicle travelling on a hill[3].

This paper shows the derivation of both the original formula and ITE's modification to the formula. Our derivation starts from the fundamental laws of physics.

Gazis, Herman and Maradudin are physicists. They intended for their formula to help traffic engineers solve one specific case in traffic flow: How long must the yellow light be to allow one particular type of driver to enter a signalized intersection legally? They did not solve the problem for every kind of driver. They solved the problem only for the case where the driver approaches the intersection at the maximum allowable speed and continues at that speed into the intersection. The formula does not work for other cases. For example it does not work for cases when drivers decelerate into the intersection. Two examples of decelerating cases are turning drivers and drivers slowing down for obstacles in front of them. For those cases, the ITE Formula yields a yellow change interval several seconds too short.

The ITE Yellow Change Interval Formula

Equation 1a is the Formula as it appears in ITE's *Traffic Engineering Handbook*[4] and *Traffic Signal Timing Manual*[5]. This Formula and its equivalents (1b, 1c) appear in traffic signal specifications for almost every jurisdiction in the world.

Equations 1. ITE Yellow Change Interval Formula

a	$Y = t_p + \left[\frac{v}{2a + 2Gg}\right]$
b	$Y = t_p + \frac{1}{2}\left[\frac{v}{a + Gg}\right]$
c	$Y = t_p + \frac{1}{2} t_b$

Variable	Description
Y	yellow light duration
t_p	perception/reaction time constant
v	vehicle's approach speed. The approach speed is not necessarily the speed limit.
a	safe comfortable deceleration of a vehicle ITE's value =10 ft/s^2 is the 50th percentile comfortable deceleration for a passenger sedan Some jurisdictions use AASHTO's value[6] = 11.2 ft/s^2 which is not comfortable deceleration but rather the 90th percentile for emergency braking.
g	Earth's gravitation acceleration constant
G	grade of the road in %/100. Downhill is negative grade.
a + Gg	effective deceleration of car
t_b	braking time. The time required by the vehicle to decelerate from **v** to a stop.

The Formula is Two Equations in One

The ITE Formula is a combination of two equations. The ITE formula embeds the equation for critical distance, then uses that result in the basic equation of motion *time = distance / velocity* in order to compute the yellow change interval.

Let us define *critical distance*. Traffic engineers define the critical distance as the distance the driver travels during the time that he perceives and reacts to the signal change to yellow plus the distance for him to brake to a stop:

Eq 2. The Critical Distance

$$c = v_0 t_p + \left[\frac{v_0^{\,2}}{2(a + Gg)}\right]$$

critical distance = perception distance + braking distance

Both perception distance and braking distance terms are basic physics formulas. The perception distance term is *rate x time = distance*. The braking distance term is little more complicated. We will show its derivation later. A crucial point to note is that v_0 is the *initial* velocity of the vehicle at the distance c upstream from the intersection. c is tied in an unbreakable physical relationship with v_0. In the braking distance term physics defines v_0 as the velocity of a vehicle where it is just beginning to brake to a stop. There is no leeway in this physical truth. v_0 cannot be measured at some arbitrary point. v_0 is not an average velocity between speed limit and intersection turning speed—that would be v_{avg}. v_0 is not the velocity measured at the stop bar—that would v_f.

In the world of transportation, v_0 must be at least the speed limit in order to give vehicles approaching the intersection at the legal speed limit the opportunity to stop. It does not matter in which lane vehicles approach. All vehicles are *allowed* to approach at the speed limit. One cannot make an exception and shorten the yellow light for turning vehicles. That would shorten the critical distance for the legally-approaching turning drivers and remove their ability to stop.

The ITE Formula embeds the critical distance. The ITE formula is the critical distance divided by the initial velocity.

Eq 3. Minimum Yellow Change Interval = Critical Distance Divided by the Initial Velocity

$$\boldsymbol{Y_{min}} = \frac{\boldsymbol{v_0\, t_p} + \left[\frac{\boldsymbol{v_0}^2}{\boldsymbol{2(a + Gg)}}\right]}{\boldsymbol{v_0}}$$

The intents of the Formula are these:

1. If the driver is farther from the intersection than the critical distance *c* when the light turns yellow, then he must stop. By embedding the braking distance into the yellow signal time, the Formula gives a faraway driver enough *distance* to stop.

2. If the light turns yellow when the driver is at c and the driver proceeds at v_0, the driver will arrive at the intersection the instant the light turns red.

3. If the driver is closer to the intersection than *c*, then the driver does not have enough *distance* to stop. The driver must proceed. The Formula gives this driver enough *time* to enter the intersection before the light turns red on the precondition that the driver approaches the intersection at a speed $\geq v_0$. The driver must proceed at the speed limit or accelerate in order to enter the intersection before the light turns red.

4. Drivers relate to this next scenario: The Formula does *not* provide drivers the *time* to decelerate while travelling inside the critical distance. Consider the driver approaching an intersection and he can no longer comfortably stop. If just then the light turns yellow, the driver will not choose to slow down. The driver knows that if he slows down, he extends the time it will take him to reach the intersection. The yellow light will run out. So he goes at least the speed limit, sometimes accelerating, hoping that he can still make it before the light turns red.

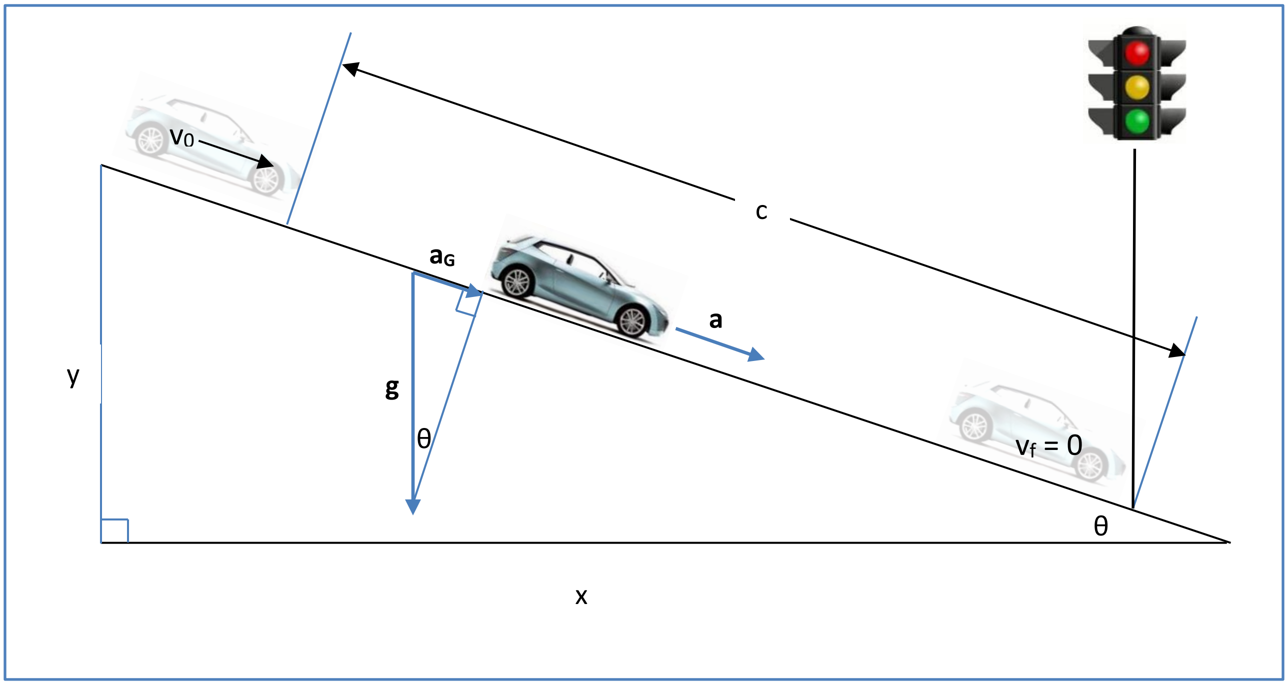
   There are dozens of common circumstances where the driver *must* slow down. Turning movements, cars pulling onto the road from side-streets, two close-by intersections, traffic congestion, hazards, railroad tracks, speed limit changes on the far side of the intersection, etc. When drivers face such circumstances, the Formula will cause them to run red lights involuntarily.

5. Assuming its preconditions are met, the ITE Formula provides the driver a legal solution to stop or go. The problem is that the reasonably perceptive driver may not know what that solution is. Drivers will still inadvertently run red lights. By setting the yellow light duration to Y_{min}, drivers do not get a margin of error. The need to stop instantly transmutes to the need to go. In order to choose correctly all the time, the Formula requires that drivers must act upon the exact position of the critical distance and the exact duration of the yellow light. Engineers do not provide drivers this information. Drivers have to estimate both distance and time and make an imprecise judgment. The zone on the road where it is not clear whether to stop or go is called an indecision zone. The indecision zone is about one hundred feet long surrounding the critical distance. When drivers are in an indecision zone, drivers tend to slam on the brakes, beat the light or run the red light inadvertently. The Formula itself, not human psychology, creates the indecision zone.

The Derivation

Consider figure 1. At the critical distance c, the driver is travelling at velocity v_0. The driver is travelling a hill of grade G. G = rise / run = y / x. Negative G values mean that the hill has a downward slope.

Fig 1. The Critical Distance with Grade



Equation	Equations 4 - 6 are Newton's Second Law with its accompanying equations of motion for acceleration and velocity. Equations 4 - 6 hold true for any object at non-relativistic speeds in reference frames in uniform motion.	
4	$\boldsymbol{F = ma}$	Newton's Second Law force = mass x acceleration
5	$\boldsymbol{a = \frac{dv}{dt}}$	acceleration = change of velocity over time
6	$\boldsymbol{v = \frac{dx}{dt}}$	velocity = displacement over time
First Assumption	The brakes of all vehicles, whether 18 wheelers to mopeds, can produce sufficient opposing force to decelerate the vehicle at "a" in all weather conditions. In physics terminology, the vehicle's brakes combined with the friction of the pavement can exert a net force **F** opposing the vehicle's motion such that the vehicle can decelerate at "a". However it is possible that the mass of the vehicle combined with the strength of its brakes and prevailing weather conditions may make it impossible for a vehicle to decelerate at "a".	
7	$\boldsymbol{c = x_p + d_b}$	c = critical distance = distance travelled during the perception/reaction time + distance travelled while braking

8	$x_p = t_p v_0$	x_p = distance travelled during perception/reaction time at initial velocity v_o.
Facts	v_o is the velocity of the vehicle at distance c Vehicles proceed at constant velocity v_o during the entire perception/reaction time.	
9	$a = \frac{dv}{dt}$	Repeat of equation 5
10	$dv = a\,dt$	Rearrange 9 algebraically
11	$\int_{v_0}^{v} dv = \int_{t_0}^{t} a(t)\,dt$	Where v_0 is the velocity at time t_0 and v is the velocity at time t.
12	$v - v_0 = at - at_0$	Integrate and evaluate 11. Assume vehicle accelerates at constant "a"— that a(t) = a.
13	$v = at + v_0$	Solve 12 for v and note that $t_0 = 0$.
14	$v = \frac{dx}{dt}$	Repeat of equation 6
15	$dx = v\,dt$	Rearrange 14.
16	$d_b = x = \int v(t)\,dt$	Integrate and solve for the braking distance displacement x.

17	$$d_b = \int_0^{t_f} (at + v_0)dt$$	Express braking distance displacement in terms of time. Substitute v(t) in equation 16 with the function for v from equation 13. d_b is the distance a vehicle travels from time 0 to time t_f. At time 0 the driver is distance d_b upstream from the intersection. At time t_f the driver arrives at the intersection.
18	$$d_b = \frac{1}{2}a{t_f}^2 + v_0 t_f$$	Evaluate 17 from t = 0 to t = t_f This is the distance a driver travels in t_f seconds starting at velocity v_0.
19	$$t_f = \frac{(v_f - v_0)}{a}$$	Rearrange 13. It takes time t_f for a vehicle to decelerate from v_o to v_f.
20	$$t_f = \frac{(-v_0)}{a}$$	v_f = 0 = the velocity of the vehicle that just stopped. v_0 is initial velocity of the vehicle before it starts accelerating.
21	$$d_b = \frac{1}{2}a(\frac{-v_0}{a})^2 + v_0(\frac{-v_0}{a})$$	Plug 20 into 18. v_0 is the velocity of the vehicle at distance d_b upstream from the intersection.

22	$$d_b = \frac{1}{2a}v_0^2 - \frac{v_0^2}{a}$$	
23	$$d_b = -\frac{v_0^2}{2a}$$	
Change of definition of acceleration	Physicists define "a" as acceleration. "a" is positive for objects increasing in speed and negative for objects decreasing in speed. Traffic engineers change the definition of "a" from acceleration to deceleration which flips the sign.	
24	$$d_b = \frac{v_0^2}{2a}$$	Driver starts at velocity v_0, decelerates at constant "a", and comes to a stop after distance d_b.
25	$$c = t_p v_0 + \frac{v_0^2}{2a}$$	The formula for the critical distance. Plug equations 8 and 23 into 7. v_0 is the velocity of the vehicle at distance c upstream from the intersection. It is the same velocity of the vehicle at distance d_b upstream from the intersection when the vehicle starts braking.
26	$$c = t_p v_0 + \frac{v_0^2}{2(a + a_G)}$$	From figure 1, modify 25 to include the component of acceleration on the vehicle due to Earth's gravity.

27	$\boldsymbol{G = y/x}$	The grade of the road = rise over run.
28	$\boldsymbol{\tan\theta = G}$	θ, the angle of the slope, has a trigonometric relationship with the road grade.
29	$\boldsymbol{\theta = \tan^{-1} G}$	Isolate θ.
30	$\boldsymbol{a_G = \mathrm{g}\sin\theta}$	From figure 1, **a_G** is the component of Earth's gravitational acceleration **g** in the direction of the vehicle's motion.
31	$\boldsymbol{a_G = \mathrm{g}\sin(\tan^{-1} G)}$	Plug 29 into 30.
32	$\boldsymbol{c = t_p v_0 + \dfrac{v_0^2}{2[a + \mathrm{g}\ \sin(\tan^{-1} G)]}}$	Plug 31 into 26. **This equation for the critical distance is precise for any grade of road.**
Assumption: The grade is small. 33	$\boldsymbol{G \approx \sin G}$ $\boldsymbol{G \approx \tan^{-1} G}$	When an angle is small, the angle is approximately equal to its sine or its arc tangent. The **small angle approximation** works for grades < 10% (0.10).
34	$\boldsymbol{c = t_p v_0 + \dfrac{v_0^2}{2[a + G\mathrm{g}]}}$	Plug in 33 into 32. This equation for the critical distance restricts itself to the acceleration effects of Earth's gravity for grades of road which

		are less than 10%.
35	$$Y_{gen} \geq \frac{t_p v_0 + \frac{v_0^2}{2[a + Gg]}}{\bar{v}}$$	This is the general form of the yellow change interval formula. $\bar{v}$ is the average velocity of the vehicle as it traverses the critical distance. Note that the formula is an inequality. The yellow change interval must be at least that of the expression on the right. The yellow change interval must be at least the time it takes a driver to traverse the critical distance. Gazis' original formula is an inequality.
35a	$$Y_{gen} \geq \frac{t_p v_0 + \frac{v_0^2}{2[a + Gg]}}{(v_0 + v_i)/2}$$ $v_0 \geq v_i$	This is another way to express equation 35. Assume a constant deceleration through the critical distance. $\bar{v}$ is the average of the velocity (v_0) at the critical distance and the velocity (v_i) at the intersection entrance point.
35b	$$Y_{gen} \geq \frac{2t_p v_0 + \frac{v_0^2}{(a + Gg)}}{(v_0 + v_i)}$$	Multiply numerator and denominator in equation 35a by 2.

35c	$$Y_{gen} \geq \frac{2t_p + \frac{v_0}{(a + \mathrm{Gg})}}{(1 + v_i/v_0)}$$	Divide numerator and denominator in equation 35b by v_0.
35d	$$Y_{gen} \geq \frac{2\left[t_p + \frac{v_0}{2(a + \mathrm{Gg})}\right]}{(1 + v_i/v_0)}$$ $$v_0 \geq \begin{cases} v_l, & v_l = \textit{speed limit} \\ v_{85}, & v_{85} > v_l \end{cases}$$ $$Y_{gen} \geq \frac{2Y}{(1 + v_i/v_0)}$$	Refactor the numerator in equation 35c. Equation 35d without the gravitation component is in Dr. Chiu Liu's paper in the *Journal of Transportation Engineering*[7]. V_{85} is the 85th percentile speed. Consider the absence of traffic signals and speed limit signs. 85% of freely flowing vehicles travel slower than this speed. 15% travel faster than this speed. Many times the speed limit is incorrectly set to less than what it is supposed to be.[8] The term in the brackets is the classic yellow change interval (equations 1, 37).
35e	$$Y_{legal} \geq \frac{2\left[t_p + \frac{v_0}{2(a + \mathrm{g}\ \sin(\tan^{-1} G))}\right]}{(1 + v_i/v_0)}$$ $$v_0 \geq \begin{cases} v_l, & v_l = \textit{speed limit} \\ v_{85}, & v_{85} > v_l \end{cases}$$	This is the yellow change interval necessary for all vehicles that decelerate at a constant rate to proceed *legally* into an intersection for road grade *g* and intersection entry velocity v_i.
Facts	1. In order for vehicles to comfortably stop, the yellow change interval has to embed a stopping distance, aka the critical distance, which is long	

	enough for vehicles to comfortably stop. 2. The critical distance is a fixed distance for all types of drivers. For example at the critical distance c upstream from the intersection, the physical characteristics of turning and straight-through drivers are the same. Both kinds of drivers need equal opportunity to stop safely and comfortably. One cannot set a turning driver's yellow change interval below that of a straight-through driver. 3. Because all the vehicles are allowed to go the speed limit, one must plug in a value for v_0 which is at least the speed limit. Anything less than the speed limit removes a driver's ability to stop from the speed limit.	
36	$$Y \geq \frac{t_p v_0 + \frac{v_0^2}{2[a + Gg]}}{v_0}$$	When a driver proceeds at the initial velocity (speed limit) from the critical distance upstream from the intersection to the intersection, the resulting yellow change interval is what traffic engineers use.
37	$$Y \geq t_p + \frac{v_0}{2[a + Gg]}$$	Simplify 36. **This is the classic yellow change interval formula.**
38	$$Y_{turn} \geq \frac{t_p v_0 + \frac{v_0^2}{2[a + Gg]}}{(v_c + 20)/2}$$	From eq. 35a, now consider a driver who needs to turn. The driver needs to slow to 20 mph to execute the turn. The driver is not in a queue. The driver sees a green or

		a flashing yellow arrow. He is going to enter the intersection at 20 mph. Formula 38 uses the model that the driver starts at velocity $\mathbf{v_c}$ at the critical distance and decelerates evenly until he is going 20 mph at the stop bar. The denominator is the equation for the driver's average speed through the critical distance. Remember that the critical distance is fixed at $\mathbf{v_0}$ (at least the speed limit) regardless of the type of driver.
39	$$Y_{extreme} \geq \frac{t_p v_0}{v_0} + \frac{\frac{v_0^2}{2[a + Gg]}}{(v_0 + 0)/2}$$	This yellow change interval formula is for an extreme U-turn. The driver nearly has to stop. The driver's final velocity is 0. This formula uses the model that driver first proceeds through the perception distance at the initial velocity and then slows at deceleration "a".
40	$$Y_{extreme} \geq t_p + \frac{v_0}{a + Gg}$$	Simplify 39. The second term is basically equation 5. It is an equation of motion. It is the physical expression for the relationship between velocity, acceleration and time. A decelerating driver needs the full time from

		Newton's Laws to decelerate. Compare equation 40 with 37: the classic yellow change interval. The classic yellow change interval effectively forbids a driver to decelerate into the intersection.

Equation 41 below is the solution. It works for all drivers for all cases. It works for straight-through movement drivers. It works for turning drivers. It works for U-turn drivers. It works for drivers at nearby intersections. It works for cases when drivers slow down for obstacles or changes in traffic flow in front of them. As opposed to the ITE yellow change interval, equation 41 always gives the driver the option to slow down without causing him to inadvertently run a red light. As opposed to the ITE yellow change interval which gives a driver half the time to stop, equation 41 gives the driver the full time to stop. The driver no longer feels the need to slam on the brakes. The ITE yellow change interval for a driver at the critical distance who sees a light turn yellow means that if the driver proceeds at the speed limit, the light turns red the instant he enters the intersection. Equation 41 for a driver who sees a light turn yellow at the critical distance means that if the driver stops, the light turns red the instant he stops at the intersection.

Eq 41. The Solution

$$Y \geq t_p + \frac{v_0}{a + gsin(\tan^{-1} G)}$$

Variable	Description
Y	duration of yellow light
t_p	perception + reaction + air-brake time
v_0	velocity of vehicle measured at $v_0^2/2[a + Gsin(tan^{-1}(g))]$ from the intersection

	$v_0 \geq$ posted speed limit, preferably the 85th percentile speed.
a	safe and comfortable deceleration The value assumes that all vehicles from motorcycles to 18-wheelers have brakes which can exert a force to decelerate the vehicle at the deceleration of ***a***.
g	Earth's gravitational constant
G	grade of road (rise over run, negative values are downhill)
$g\sin(\tan^{-1}(G))$	precise expression for the contribution of Earth's gravity towards a vehicle's deceleration on a hill of grade G. When $G < 0.10$, $gG \approx g\sin(\tan^{-1}(G))$. When $G > 0$, set G to 0. It is wrong to decrease deceleration when travelling up a hill for two reasons: (1) Adding gG to deceleration assumes emergency braking. The gG term comes from the misapplication of the stopping sight distance formula (SSD) to yellow change intervals. The SSD mathematically expresses the emergency braking condition. It is the case where the coefficient of friction between pavement and tire has reached its maximum. The math to describe that is the addition of gG. But emergency braking is not when approaching an intersection comfortably. (2) When a vehicle ascends a hill under normal conditions, gravity slows the vehicle, decreasing its average velocity through the critical distance and therefore requiring the addition of length to the yellow change interval.[9]

Authors

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